1.0

4.5
5.0
5.6
6.1

2.8 2.5

3.2 2.2

3.6

4.0 2.0

1.1

1.8

1.25 1.4 1.6

MICROCOPY RESOLUTION TEST CHART

NATIONAL BUREAU OF STANDARDS-1963-A

DEPARTMENT OF ELECTRICAL ENGINEERING
TEXAS TECH UNIVERSITY
AND
DEPARTMENT OF ELECTRICAL ENGINEERING
OLD DOMINION UNIVERSITY

LEVEL *II*

RESULTS ON STATISTICAL PERFORMANCE ANALYSIS OF

CONTROL AND ESTIMATING SYSTEMS: A SUMMARY

D D C

NOV 20 1978

F

By

Stanley R. Liberty

Final Report

This document has been approved
for public release and sale; its
distribution is unlimited.

Prepared for the
Office of Naval Research
800 North Quincy Street
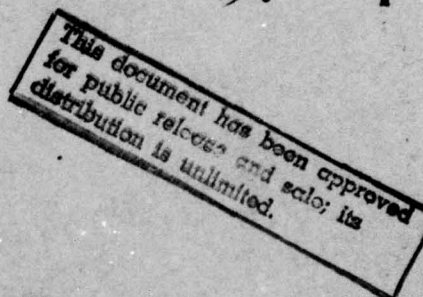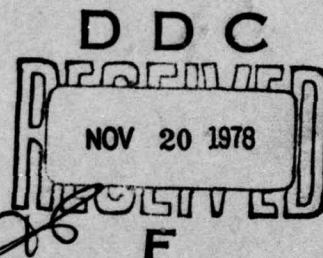Arlington, Virginia

Under
Contract Nos. N00014-73-A-0434-0002
N00014-75-C-0779

Submitted by
Texas Tech University
Lubbock, Texas
(Prime Contractor)
and
Old Dominion University Research Foundation
Norfolk, Virginia
(Subcontractor)

July 1978   78 11 20 057

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE *(and Subtitle)* Results on Statistical Performance Analysis of Control and Estimating Systems: A Summary,		5. TYPE OF REPORT & PERIOD COVERED Final rept,
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Stanley R. Liberty		8. CONTRACT OR GRANT NUMBER(s) N00014-73-A-0434-0002, N00014-75-C-0779
9. PERFORMING ORGANIZATION NAME AND ADDRESS Texas Tech University Department of Electrical Engineering Lubbock, TX 79409		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS RR021-05-01
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research 800 North Quincy Arlington, VA 22217		12. REPORT DATE July 1978
		13. NUMBER OF PAGES 74
14. MONITORING AGENCY NAME & ADDRESS*(if different from Controlling Office)*		15. SECURITY CLASS. *(of this report)* UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT *(of this Report)*

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT *(of the abstract entered in Block 20, if different from Report)*

18. SUPPLEMENTARY NOTES

19. KEY WORDS *(Continue on reverse side if necessary and identify by block number)*

Linear Quadratic Gaussian
Performance Analysis
Cumulants

20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

This report is a summary of research performed under ONR Contract N00014-75-C-0779. Two works previously not reported are appended for ease of reference and contain new results on control and estimator design according to cumulant based performance criteria.

RESULTS ON STATISTICAL PERFORMANCE ANALYSIS OF

CONTROL AND ESTIMATING SYSTEMS: A SUMMARY*

S. R. Liberty**

Department of Electrical Engineering

Texas Tech University

Lubbock, Texas 79409

TABLE OF CONTENTS

I. SUMMARY

Research performed under Contract N00014-75-C-0779 was completed March 3, 1978. This research has explored probabilistic and statistical structures of the Linear-Quadratic-Gaussian class of control and estimating systems for purposes of development and enhancement of design procedures. En route a complete theory of "performance analysis" has been developed for this problem class.

Performance analysis allows the system designer to see the quality of a given design in a complete statistical sense. As such, it becomes part of a computer-aided design procedure but, unfortunately, does not explicitly tell the designer how to modify the design to achieve better performance. Thus, the research effort has recently focused on the "selection" aspect of design.

A complete theory of control selection has not been established in this research effort; however, significant progress has been made toward a complete theory. In particular, formulas for statistics of LQG design performance measures for both continuous and discrete time systems have been derived. Also, a control selection technique based upon this formulation has been developed. This control selection technique, when coupled with the performance analysis technique, becomes an effective design procedure for stochastic control systems. An inherent property of the resulting designs is that they all contain estimators in their feedback structures and exhibit a separation property.

Two interim technical reports were distributed on the performance analysis and control selection aspects. These were:

[1] S. R. Liberty and R. C. Hartwig, "On the Essential Quadratic Nature of LQG Control-Performance Measure Cumulants", Interim Technical Report-Contract N00014-75-C-0779, Texas Tech University, August 1975, and

[2] R. C. Hartwig, "Design of Stochastic Linear Control Systems According to Cumulant Based Performance Criteria", Interim Technical Report-Contract N00014-75-C-0779, Texas Tech University, May 1976.

Report [1] also appeared under the same title in

[3] Information and Control, Vol. 32, No. 3, November 1976.

Work related to [1] and portions of [2] have appeared, or will appear, in

[4] J. M. Brown, R. C. Hartwig and S. R. Liberty, "On the Design and Performance Analysis of Linear Stochastic Regulators", Proc. of the 7th Asilomar Conference on Circuits and Systems, pp. 563-566, November 1973,

[5] R. C. Hartwig and S. R. Liberty, "A New Class of Statistical Performance Criteria for Stochastic Linear Control Systems", Proc. of the IEEE Conference on Decision and Control, pp. 1136-1141, November 1976,

[6] S. R. Liberty, "Performance Analysis of Stochastic Linear Control Systems: A New Viewpoint", Proc. of the International Forum on Alternatives for Multivariable Control, pp. 79-86, October 1977,

and

CONTENTS:

[7] S. R. Liberty and R. C. Hartwig, "Design-Performance-Measure Statistics for Stochastic Linear Control Systems", to appear in IEEE Transactions on Automatic Control, Vol. AC-23, No. 6, December 1978,

which is included here as Appendix A.

Also, the following theses on these topics have been completed:

[8] R. C. Hartwig, "Cumulants of an IQF Differential Equations", Master of Science Thesis, Texas Tech University, December 1973,

[9] J. M. Brown, "A Computer-Aided Design-Analysis Technique for Stochastic Linear Regulator Problems", Master of Science Thesis, Texas Tech University, August 1973,

and

[10] R. C. Hartwig, "Design of Stochastic Linear Control Systems According to Cumulant Based Performance Criteria", Ph.D. Dissertation, Texas Tech University, May 1976.

Another thesis was completed on additional topics involving Riccati equations arising in cumulant based design procedures for both control and estimating systems. This work,

and

[11] M. L. Yao, "Initial Studies of Riccati Equations Arising in Stochastic Linear System Theory", Master of Science Thesis, Texas Tech University, December 1977,

is included here as Appendix B. A second Ph.D. dissertation on optimal cumulant control selection in discrete time systems is in the final stages of preparation and will appear as an interim report on Contract N00014-78-C-0443.

2

II. APPENDIX A

DESIGN-PERFORMANCE-MEASURE STATISTICS FOR

STOCHASTIC LINEAR CONTROL SYSTEMS*

STANLEY R. LIBERTY, Member, IEEE, and

RONALD C. HARTWIG, Member, IEEE

ABSTRACT - Formulas for statistics of the standard integral
quadratic performance measure used in stochastic
linear control system design are derived. The
formulas are expressed in terms of dynamical
variables under the usual assumptions on noise,
plant, and admissibility of control. All of
these dynamical variables are expressed as
linear transformations of plant state estimates.
The practicality of this work, which is directed
toward the establishment of new, statistically
based design procedures for stochastic linear
systems, is demonstrated by example.

S. R. Liberty is with the Department of Electrical Engineering,
Old Dominion University, Norfolk, Virginia 23508.

R. C. Hartwig is with Department 4361, Sandia Laboratories,
Albuquerque, New Mexico 87115.

I. INTRODUCTION

Once a state model for a stochastic linear system is obtained, the analytical aspects of designing a feedback controller can be conceptually dichotomized. We refer to the first part as "control selection" and to the second as "performance analysis."

In the control selection part the designer chooses a performance measure that reflects a priori design specifications. Statistical indices associated with this performance measure are then specified and, finally, a controller is attained via optimization of these indices.

Performance analysis involves determination of how a specific controller behaves. This is accomplished by computing statistical or probabilistic descriptions of performance measures that, in general, are different than the performance measure chosen in the control selection part of the design procedure. For example, these measures may be selected to give the designer specific insight into how well the controller is regulating or how much control effort is being expended.

For certain classes of stochastic linear control systems the performance analysis problem has been solved; see [7] and [14]. However, the control selection aspect of the design problem has only been solved in special cases. For the class of systems treated in this paper, namely the Linear-Quadratic-Gaussian (LQG) class, a feedback controller that minimizes the mean of an integral quadratic performance measure

-1-

has been found [16]. An open-loop controller that minimizes the variance of an integral quadratic performance measure subject to a constraint on its mean has also been discovered [9], [10], [12], [13], but feedback solutions in the context of performance measure statistics beyond the mean have not surfaced. The reason for this is a lack of tractable higher order statistical descriptions of the performance measure.

In this paper we present the formulation of a complete set of statistics of the integral quadratic performance measure normally employed in stochastic linear control system design. These results are an outgrowth of work initiated a little over ten years ago by Sain [11]. Our formulation is expressed in terms of dynamical variables related to an estimate of the plant state and should lead the way to new classes of feedback control structures in the stochastic linear context.

We have attempted to keep our notation as compatible as possible with that in the tutorial paper by Tse [15] and strongly recommend that this reference, along with [8], be primarily used by the reader. Our presentation is formal so no special mathematical skills should be required for a fundamental understanding of the material.

In Section II we describe the system, the performance measure and the control objective. Section III contains the development of a complete statistical description of this performance. An example of control system design utilizing the new formulation is presented in Section IV.

II. THE SYSTEM DESCRIPTION AND PERFORMANCE

Let R^p denote the p-fold Cartesian product of the real line, and let I denote the real line interval $[t_o, t_f]$. We wish to control the noisy linear system described on I by

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t) + \xi(t) \tag{1}$$

and

$$z(t) = C(t)x(t) + \theta(t), \tag{2}$$

where the state $x(t) \in R^n$, the control action $u(t) \in R^m$, and the observation $z(t) \in R^r$. The initial condition for (1), $x(t_o)$, is assumed to be Gaussian with mean

$$x_o = E\{x(t_o)\} \tag{3}$$

and covariance

$$\Sigma_o = E\{[x(t_o)-x_o][x^T(t_o)-x_o^T]\} \tag{4}$$

where $(^T)$ denotes matrix transposition. The state process noise, ξ, and the observation noise, θ, are zero-mean Gaussian-white with

$$E\{\xi(t)\theta^T(\tau)\} = 0, \qquad\qquad t,\tau \in I, \tag{5}$$

$$E\{[x(t_o)-x_o]\xi^T(t)\} = 0, \qquad\qquad t \in I, \tag{6}$$

$$E\{[x(t_o)-x_o]\theta^T(t)\} = 0, \qquad\qquad t \in I, \tag{7}$$

$$E\{\xi(t)\xi^T(\tau)\} = \Xi(t)\delta(t-\tau), \qquad\qquad t,\tau \in I, \tag{8}$$

and

$$E\{\theta(t)\theta^T(\tau)\} = \Theta(t)\delta(t-\tau), \qquad\qquad t,\tau \in I, \tag{9}$$

-3-

where $\Xi(t)$ and $\Theta(t)$ are positive semi-definite and positive definite, respectively, on I.

We require that the control action, u, be a causal function of the observation. That is,

$$u(t) = \psi(t, z(\tau):\tau\varepsilon[t_o,t)) \qquad\qquad t\varepsilon I, \qquad (10)$$

where ψ satisfies certain technical assumptions stated in [15]; also see [16]. All matrix functions on I and the mapping ψ are assumed to be smooth enough to guarantee mean-square continuity of the state process on I.

For design purposes we define a "design-performance-measure"

$$J \triangleq x^T(t_f)Sx(t_f) + \int_{t_o}^{t_f} [x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)]dt, \qquad (11)$$

where the terminal penalty weighting, S, is symmetric and positive semi-definite as is the weighting $Q(t)$ on I. The weighting $R(t)$ is symmetric and positive definite on I, and both $Q(t)$ and $R(t)$ are continuous on I. These weighting matrices are given values by the designer that reflect a priori design specifications involving the relative importance of state regulation and control effort. The design objective is to choose u in (10) so system performance is "good" in some sense.

The functional, J, assigns a non-negative real number to each sample run of the control system with small values implying good performance, However, the question of quality of performance is multiply clouded. First, J is random so it is only meaningful to refer to J in a statistical or probabilistic sense. Second, since J is the sum of terms representing measures of state regulation

and control effort, the individual quality of these measures is not apparent in a broad statistical description of J. We will not address these subtleties here, but will concentrate on obtaining a statistical description of J that might be used as a basis for selection of control action. In Section IV, the concepts of control selection and performance analysis are demonstrated by example.

III. A COMPLETE STATISTICAL DESCRIPTION OF J

Let F_σ be the sigma-algebra induced by the observation $\{z(\tau): \tau \varepsilon[t_o, \sigma)\}$.
When $\sigma = t_f$ we will write F without a subscript. We will now generate con-
ditional statistics of J . Expand the process modeled by (1) in an orthonormal
series,

$$x(t) \sim \sum_{i=1}^{\infty} x_i \phi_i(t), \qquad t\varepsilon I , \qquad (12)$$

where the x_i's are scalar random variables given by

$$x_i = x^T(t_f) S\phi_i(t_f) + \int_{t_o}^{t_f} x^T(t)Q(t)\phi_i(t)dt, \qquad \forall i, \qquad (13)$$

and the orthonormality constraint on the nonrandom, vector-valued ϕ_i's is
given by

$$\phi_i^T(t_f)S\phi_j(t_f) + \int_{t_o}^{t_f}\phi_i^T(t)Q(t)\phi_j(t)dt = \delta_{ij} , \qquad \forall i, j. \qquad (14)$$

In addition, we require that the x_i's be conditionally uncorrelated, that is

$$E\{[x_i - m_i][x_j - m_j]|F\} = \lambda_i \delta_{ij}, \qquad \forall i, j, \qquad (15)$$

where m_i is the conditional mean of x_i given by

$$m_i = E\{x^T(t_f)|F\} S\phi_i(t_f)$$

$$+ \int_{t_o}^{t_f} E\{x^T(t)|F\}Q(t)\phi_i(t)dt, \qquad \forall i, j. \qquad (16)$$

-6-

A necessary and sufficient condition for (15) given (13) and (14) is that ϕ_i and λ_i satisfy

$$\lambda_i \phi_i(t) = \int_{t_o}^{t_f} \Gamma(t,\tau)Q(\tau)\phi_i(\tau)d\tau + \Gamma(t,t_f) S\phi_i(t_f), \quad t \varepsilon I, \quad \forall i, \qquad (17)$$

where Γ is the smoothed-estimate, error-covariance kernel of the state process. That is, let the smoothed estimate of $x(t)$ be denoted by

$$\hat{x}(t|t_f) = E\{x(t)|F\}, \quad t \varepsilon I. \qquad (18)$$

Then Γ is given by

$$\Gamma(t,\tau) = E\{[x(t) - \hat{x}(t|t_f)] [x^T(\tau) - \hat{x}^T(\tau|t_f)]|F\} \quad t,\tau \varepsilon I. \qquad (19)$$

As a consequence of the linear-Gaussian assumptions of Section II and the technical assumptions on ψ,

$$\Gamma(t,\tau) = E\{\Gamma(t,\tau)\} . \qquad (20)$$

That is, Γ is nonrandom implying that each λ_i is nonrandom.

Under the assumptions we have made, J is finite with probability one; see Doob [1]. It follows that the series in (12) converges in the square integrable sense. Thus,

$$J = \sum_{i=1}^{\infty} x_i^2 + \int_{t_o}^{t_f} u^T(t)R(t)u(t)dt, \qquad (21)$$

with probability one; see Kolmogorov and Fomin [5].

-7-

Since x is conditionally Gaussian each x_i is conditionally Gaussian. But, we have forced the x_i's to be conditionally uncorrelated, thus they are conditionally independent as are their squares. The conditional characteristic function of each x_i^2 term in (21) is of the noncentral chi-square type given by,

$$C_{x_i^2|F}(j\omega) = (1-2j\omega\lambda_i)^{-\frac{1}{2}}\exp[j\omega m_i^2(1-2j\omega\lambda_i)^{-1}].$$ (22)

The conditional characteristic function of J follows as,

$$C_{J|F}(j\omega) = [\prod_{i=1}^{\infty}(1-2\omega\lambda_i)^{-\frac{1}{2}}] \cdot \exp[j\omega \int_{t_0}^{t_f} u^T(t)R(t)u(t)dt$$

$$+ \sum_{i=1}^{\infty} j\omega m_i^2(1-2j\omega\lambda_i)^{-1}].$$ (23)

In our previous work [6], [7] we have observed that in the Linear-Quadratic-Gaussian class of systems the second characteristic function generates tractable statistical forms. The second conditional characteristic function, $T_{J|F}$, is defined as the natural logarithm of $C_{J|F}$, that is,

$$T_{J|F}(j\omega) \stackrel{\Delta}{=} \ln[C_{J|F}(j\omega)].$$ (24)

The formal MacLaurin series representation of $T_{J|F}$ is

$$T_{J|F}(j\omega) = \sum_{i-1}^{\infty} \kappa_{i|F} \frac{(j\omega)^i}{i!}$$ (25)

where the coefficients $\kappa_{i|F}$ are called conditional cumulants. Utilizing (23), it can be easily shown that the first conditional cumulant of J is given by

-8-

$$\kappa_{1|F} = \sum_{i=1}^{\infty} \lambda_i + \sum_{i=1}^{\infty} m_i^2 + \int_{t_o}^{t_f} u^T(t)R(t)u(t)dt, \tag{26}$$

while the remaining conditional cumulants are of the form

$$\kappa_{k|F} = (k-1)!2^{k-1} \sum_{i=1}^{\infty} \lambda_i^k + k!2^{k-1} \sum_{i=1}^{\infty} m_i^2 \lambda_i^{k-1}, \quad k > 1. \tag{27}$$

Although the conditional cumulants as given by (26) and (27) are complete in the sense that any statistic of J can be derived from them, they are not in an attractive form for the control system designer since they are not expressed in terms of system variables. To accomplish this we must attack the series expressions in (26) and (27). The first step is to define "iterated kernels"

$$\Gamma^{(1)}(t,\tau) \triangleq \Gamma(t,\tau) \tag{28}$$

and

$$\Gamma^{(k)}(t,\tau) \triangleq \Gamma(t, t_f) S\Gamma^{(k-1)}(t_f,\tau)$$

$$+ \int_{t_o}^{t_f} \Gamma(t,\sigma) Q(\sigma)\Gamma^{(k-1)}(\sigma,\tau) d\sigma, \quad k > 1. \tag{29}$$

It can be inductively shown using (14) and (17) that

$$\Gamma^{(k)}(t,\tau) = \sum_{i=1}^{\infty} \lambda_i^k \phi_i(t)\phi_i^T(\tau). \tag{30}$$

It follows that the expression, $\sum_{i=1}^{\infty} \lambda_i^k$, can be written as

$$\sum_{i=1}^{\infty} \lambda_i^k = \text{Tr}[S\Gamma^{(k)}(t_f,t_f) + \int_{t_o}^{t_f} Q(t)\Gamma^{(k)}(t,t) dt], \quad k \geq 1, \tag{31}$$

-9-

where $\text{Tr} \, [\cdot]$ denotes the trace of the enclosed matrix. Utilizing (16), (18) and (30) it follows that

$$\sum_{i=1}^{\infty} m_i^2 \lambda_i^{k-1} = \hat{x}^T(t_f|t_f) \, S\Gamma^{(k-1)}(t_f,t_f) \, S\hat{x}(t_f|t_f)$$

$$+ \hat{x}^T(t_f|t_f)S \int_{t_o}^{t_f} \Gamma^{(k-1)}(t_f,t)Q(t)\hat{x}(t|t_f) \, dt$$

$$+ \int_{t_o}^{t_f} \hat{x}(t|t_f)Q(t) \, \Gamma^{(k-1)}(t, t_f)dt \, S\hat{x}(t_f|t_f)$$

$$+ \int_{t_o}^{t_f} \int_{t_o}^{t_f} \hat{x}^T(t|t_f)Q(t)\Gamma^{(k-1)}(t,\tau)Q(\tau)\hat{x}(\tau|t_f)d\tau dt, \quad k>1. \quad (32)$$

For the case, $k=1$, it is easily seen that

$$\sum_{i=1}^{\infty} m_i^2 = \hat{x}^T(t_f|t_f)S\hat{x}(t_f|t_f)$$

$$+ \int_{t_o}^{t_f} \hat{x}^T(t|t_f)Q(t)\hat{x}(t|t_f)dt. \quad (33)$$

Consider the last term in (32) and note that it contains a symmetric (in argument) integrand. Therefore it may be rewritten as

$$\int_{t_o}^{t_f} \int_{t_o}^{t_f} \hat{x}^T(t|t_f)Q(t)\Gamma^{(k-1)}(t,\tau)Q(\tau)\hat{x}(\tau|t_f)d\tau dt$$

$$= 2 \int_{t_o}^{t_f} \hat{x}^T(t|t_f)Q(t) \int_{t_o}^{t} \Gamma^{(k-1)}(t,\tau)Q(\tau)\hat{x}(\tau|t_f)d\tau dt \tag{34}$$

Define the new n-vector valued variable

$$\eta_{k-1}(t) \triangleq \int_{t_o}^{t} \Gamma^{(k-1)}(t,\tau)Q(\tau)\hat{x}(\tau|t_f)d\tau \tag{35}$$

and note that

$$\eta_{k-1}(t_o) = 0. \tag{36}$$

The conditional cumulants can now be written as

$$\kappa_{1|F} = \hat{x}^T(t_f|t_f) S\hat{x}(t_f|t_f) + \int_{t_o}^{t_f}[\hat{x}^T(t|t_f)Q(t)\hat{x}(t|t_f)$$

$$+ u^T(t)R(t)u(t)]dt \tag{37}$$

$$+ \text{Tr}[S\Gamma(t_f,t_f) + \int_{t_o}^{t_f} Q(t)\Gamma(t,t)dt],$$

and

$$\kappa_{k|F} = k!2^{k-1}[\hat{x}^T(t_f|t_f)S\Gamma^{(k-1)}(t_f,t_f)S\hat{x}(t_f|t_f)$$

$$+ 2\hat{x}^T(t_f|t_f)S\eta_{k-1}(t_f) + 2\int_{t_o}^{t_f} \hat{x}^T(t|t_f)Q(t)\eta_{k-1}(t)dt]$$

$$+ (k-1)!2^{k-1}\text{Tr}[S\Gamma^{(k)}(t_f,t_f) + \int_{t_o}^{t_f} Q(t)\Gamma^{(k)}(t,t)dt], \quad k > 1. \tag{38}$$

-11-

Equations (37) and (38) provide us with the conditional cumulants of the design-performance measure, J, expressed in terms of the smoothed estimate of the x-process and the corresponding error covariance kernel. Any statistic of J can be expressed in terms of the conditional cumulants via their relationship to conditional moments. This relationship is of the same form as (42) below. For example, denoting a cumulant of J by κ_k, we have

$$E\{J\} = \kappa_1 = E\{\kappa_{1|F}\},$$ (39)

$$Var\{J\} = \kappa_2 = E\{\kappa_{2|F}\} + Var\{\kappa_{1|F}\},$$ (40)

and in general

$$\kappa_k = E\{\kappa_{k|F}\} + \{\text{statistics of lower order conditional cumulants}\}.$$ (41)

The relationship between noncentral moments and cumulants is well tabulated [4]. Denoting a noncentral moment of J by μ_k, this relationship is given by

$$\mu_{k+1} = \sum_{j=0}^{k} \binom{k}{j} \mu_{k-j} \kappa_{j+1}.$$ (42)

Those familiar with the traditional minimum mean LQG problem may be a little suspicious of equation (37) since it is well known that $E\{\kappa_{1|F}\}$ is normally expressed in terms of the filtered estimate [3], $\hat{x}(t|t)$, and its corresponding error covariance with precisely the same structure as (37) under expectation; see [15]. To demonstrate the equivalence of these two formulations note that in view of (10) the smoothed estimate can be expressed as

$$\hat{x}(t|t_f) = \hat{x}(t|t) + \int_t^{t_f} K(t,\tau) \, C^T(\tau)\Theta^{-1}(\tau) \, C(\tau) \, \nu(\tau|\tau)d\tau, \; t\varepsilon I, \quad (43)$$

see [8], where

$$\hat{x}(t|t) = E\{x(t)|F_t\}, \quad (44)$$

$$K(t,\tau) = E\{[x(t) - \hat{x}(t|t)][x^T(\tau) - \hat{x}^T(\tau|\tau)]\}, \quad (45)$$

and the "innovation" [2], ν, is given by

$$\nu(t|t) = C(t)[x(t) - \hat{x}(t|t)] + \theta(t). \quad (46)$$

The smoothed error covariance can also be expressed as

$$\Gamma(t,\tau) = K(t,\tau) - \int_{tv\tau}^{t_f} K(t,\sigma)C^T(\sigma)\Theta^{-1}(\sigma) \, C(\sigma) \, K(\sigma,\tau)d\sigma \quad (47)$$

where $tv\tau$ means max $[t,\tau]$.

Substitution of (43) and (47) into (37) and application of expectation immediately yields

$$\kappa_1 = E\{\kappa_{1|F}\} = E\{\hat{x}^T(t_f|t_f)S\hat{x}(t_f|t_f) + \int_{t_o}^{t_f}[\hat{x}^T(t|t)Q(t)\hat{x}(t|t)$$

$$+ u^T(t)R(t)u(t)]dt\} + Tr[S\Sigma(t_f) + \int_{t_o}^{t_f} Q(t)\Sigma(t)dt], \quad (48)$$

where
$$\Sigma(t) \triangleq K(t,t) \quad (49)$$

-13-

and we have utilized the fact that ν is white noise with covariance

$$E\{\nu(t|t)\nu^T(\tau|\tau)\} = \Theta(t)\delta(t-\tau). \tag{50}$$

In deriving (48) from (37) only one subtlety arises that might be troublesome to the reader. In particular, two terms of the form

$$E\{\int_{t_o}^{t_f} \hat{x}^T(t|t)Q(t) \int_t^{t_f} K(t,\tau)C^T(\tau)\Theta^{-1}(\tau)C(\tau)\nu(\tau|\tau)dt\}$$

$$= Tr[\int_{t_o}^{t_f} Q(t) \int_t^{t_f} K(t,\tau)C^T(\tau)\Theta^{-1}(\tau)C(\tau)E\{\nu(\tau|\tau)\hat{x}^T(t|t)\}d\tau dt] \tag{51}$$

arise.

It is well known [14] that under some technical assumptions on the causal mapping ψ in (10)

$$\hat{x}(t|t) = \int_{t_o}^t G(t,\tau)\nu(\tau|\tau)d\tau + \int_{t_o}^t \Phi(t,\tau)B(\tau)u(\tau)d\tau + \Phi(t,t_o)x_o \tag{52}$$

for some kernel G where Φ is the transition matrix associated with A in (1). Since the control u is assumed to be a causal function of the observation z, which in turn can be expressed as a causal function of the innovation, ν, it follows in view of (50) that the innermost integrand in (51) is zero almost everywhere. Consequently, the terms in question vanish under expectation.

IV. A DESIGN EXAMPLE

How might these formulations be utilized? There are many possible
answers to this question. For example, let us assume that the design
objective is to select a controller that will keep the variance of J
small without the mean of J becoming too large. Such an objective
suggests selection of a weighted sum of the indices, mean and variance,
for optimization. Such a criterion is:

$$\text{minimize} \quad [E\{\kappa_1|F\} + \alpha(E\{\kappa_2|F\} + \text{Var}\{\kappa_1|F\})]$$

over all admissible control laws subject to the obvious dynamical con-
straints. The nonnegative real parameter α allows a trade-off between mean
and variance to be effected in the design procedure. For example, when α
is zero the criterion collapses to the traditional minimum mean criterion.
When the designer selects large α, emphasis is placed upon making the vari-
ance of performance small.

The design procedure consists of selection of several controllers via
the above criterion, with performance analysis of each until an acceptable
trade-off between mean and variance of performance is achieved.

Unfortunately, research on this class of problems has not evolved to
the point where the complete solution is known. We can, however, modify
the criterion to yield a classical problem formulation that will lead to
an interesting class of feedback control laws.

To accomplish this we retain only second degree terms in the criterion.
Thus, the term $\text{Var}\{\kappa_1|F\}$ is arbitrarily dropped. Next we substitute (43)
and (47) into (35) and (38) for $k = 2$ and fully expand the criterion. All
terms that are not affected by control action or involve future dynamical

-15-

operations are discarded to allow simple enforcement of (10). For simplicity, terms containing the integral expression in (47) are also discarded.

It may appear that this surgery is rather drastic but actually much of the original objective has been retained in the modified criterion which now looks like

$$\min_{u} E\{\hat{x}^T(t_f|t_f)S\hat{x}(t_f|t_f) + \int_{t_o}^{t_f} [\hat{x}^T(t|t)Q(t)\hat{x}(t|t) + u^T(t)R(t)u(t)]dt$$

$$+ 4\alpha\hat{x}^T(t_f|t_f)S\Sigma(t_f)S\hat{x}(t_f|t_f) + 8\alpha\hat{x}^T(t_f|t_f)S\eta(t_f)$$

$$+ 8\alpha \int_{t_o}^{t_f} \hat{x}^T(t|t)Q(t)\hat{\eta}(t)dt\},$$

where

$$\frac{d}{dt}\hat{\eta}(t) = [A(t) - \Sigma(t)C^T(t)\Theta^{-1}(t)C(t)]\hat{\eta}(t) + \Sigma(t)Q(t)\hat{x}(t|t), \qquad (53)$$

with

$$\hat{\eta}(t_o) = 0. \qquad (54)$$

Define the augmented matrices

$$\bar{x}(t) \triangleq \begin{bmatrix} \hat{x}(t|t) \\ \hat{\eta}(t) \end{bmatrix}, \qquad (55)$$

$$\bar{Q}(t) \triangleq \begin{bmatrix} Q(t) & 4\alpha Q(t) \\ 4\alpha Q(t) & 0 \end{bmatrix}, \qquad (56)$$

$$\bar{S} \triangleq \begin{bmatrix} S + 4\alpha S\Sigma(t_f)S & 4\alpha S \\ 4\alpha S & 0 \end{bmatrix}, \qquad (57)$$

$$\overline{A}(t) \triangleq \begin{bmatrix} A(t) & 0 \\ \Sigma(t)Q(t) & A(t) - \Sigma(t)C(t)\Theta^{-1}(t)C(t) \end{bmatrix} \tag{58}$$

$$\overline{B}(t) \triangleq \begin{bmatrix} B(t) \\ 0 \end{bmatrix} , \tag{59}$$

and

$$\overline{W}(t) \triangleq \begin{bmatrix} W(t) \\ 0 \end{bmatrix} , \tag{60}$$

where

$$W(t) \triangleq \Sigma(t)C^T(t)\Theta^{-1}(t) \tag{61}$$

The criterion can then be rewritten as

$$\min E\{ \overline{x}^T(t_f)\overline{S}\overline{x}(t_f) + \int_{t_o}^{t_f} [\overline{x}^T(t)\overline{Q}(t)\overline{x}(t) + u^T(t)R(t)u(t)]dt\}$$

over admissible u subject to

$$\dot{\overline{x}}(t) = \overline{A}(t)\overline{x}(t) + \overline{B}(t)u(t) + \overline{W}(t)\nu(t|t), \tag{62}$$

with

$$\overline{x}(t_o) = \begin{bmatrix} x_o \\ 0 \end{bmatrix} . \tag{63}$$

The solution of this "accessible state" problem is, of course, well known [15] and given by

$$u(t) = -R^{-1}(t)\overline{B}^T(t)M(t)\overline{x}(t), \qquad t\epsilon I, \tag{64}$$

where the 2nx2n matrix $M(t)$ is the solution on I of

$$\frac{d}{dt}M(t) = -M(t)\overline{A}(t) - \overline{A}^T(t)M(t) - \overline{Q}(t) + M(t)\overline{B}(t)R^{-1}(t)\overline{B}^T(t)M(t), \tag{65}$$

with

$$M(t_f) = \overline{S}. \tag{66}$$

-17-

A Simple Example: Consider the scalar system described on [0,1] by

$$\dot{x} = x + u + \xi \tag{1'}$$

and

$$z = x + \theta \tag{2'}$$

where

$$x_o = 1 \tag{3'}$$

and

$$\Sigma_o = 0. \tag{4'}$$

Let conditions (5) through (7) hold with

$$E\{\xi(t)\xi^T(\tau)\} = .25\delta(t-\tau) \tag{8'}$$

and

$$E\{\theta(t)\theta^T(\tau)\} = .35\delta(t-\tau). \tag{9'}$$

Select the design-performance-measure

$$J = \int_0^1 [x^2(t) + u^2(t)]dt \tag{11'}$$

and calculate, via (65) and (64), the feedback controller that optimizes the modified criterion for several values of α. For each of these controllers carry out performance analysis by computing statistical and probabilistic descriptions of state regulation and control effort. Specifically, obtain such descriptions for the function-space squared-norm of the plant state trajectory and the control action trajectory. That is, select post design performance measures

$$J_x = \int_0^1 x^2(t)dt \tag{67}$$

and

$$J_u = \int_0^1 u^2(t)dt. \tag{68}$$

-18-

The mean M, and variance, V, of J, J_x and J_u are plotted versus $\propto$ in Figures 1-3 respectively. Note that the original objective of making the variance of J small at the expense of larger mean is achieved. Also note that both mean and variance of plant state regulation are improved by selecting controllers with large $\propto$ values. This "good" regulation is paid for by a corresponding increase in the mean and variance of control effort.

Figures 4, 5, and 6 contain probability densities for J, J_x, and J_u respectively that provide a complete statistical picture of performance.

Some observations should be noted. The class of control laws generated by the criterion is truly dynamical in that the feedback law contains $\hat{n}$ dynamics driven by the filtered estimate of the plant state. A separation property is inherent. The extension of the traditional LQG criterion to include second order statistical terms is simply parameterized. By incorporating "performance analysis" techniques from [7] into the design procedure, insight into controller properties is enhanced.

Figure 2. Mean and Variance of J_x versus α.
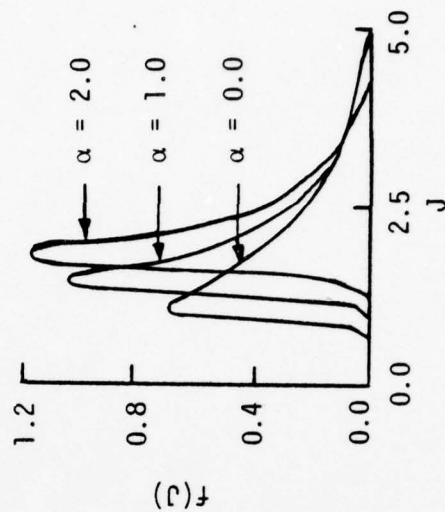
Figure 4. Probability Densities of J.

Figure 1. Mean and Variance of J versus α.

Figure 3. Mean and Variance of J_u versus α.

-20-

Figure 6. Probability Densities of J_u.

Figure 5. Probability Densities of J_x.

V. CONCLUSIONS

A great deal of research remains to be done. We have only scratched
the surface. In particular, for cumulants of performance beyond the mean,
it is not apparent that the smoothed estimate formulation will collapse
to a filtered estimate formulation as it did in (48). The resulting pres-
ence of noncausal variables is troublesome in control selection and moti-
vated our approach to the example of Section IV.

Consequently, we have not presented a complete theory and feel that
there may be a better formulation of this problem class than that pre-
sented here. Despite this, we are encouraged by these results since they
demonstrate a richness of the LQG problem class that was not apparent in
the minimum mean results of the previous decade.

The practical value of the design viewpoint expressed in Section IV
should be apparent. This viewpoint can, of course, be directly extended
to include higher order statistics with the obvious consequent increase
in off-line computation and complexity of feedback structure.

ACKNOWLEDGEMENTS

REFERENCES

[1]   J. L. Doob, *Stochastic Processes*. New York: Wiley, 1953.

[2]   T. Kailath, "An Innovations Approach to Least-Squares Estimation
        Part I: Linear Filtering in Additive White Noise," IEEE Trans.
        on Automat. Contr., vol. AC-13, no. 6, pp. 646-654, December 1968.

[3]   R. E. Kalman and R. S. Bucy, "New Results in Linear Filtering and
        Prediction Theory," J. of Basic Engrg., vol. 83, pp. 95-107,
        March 1961.

[4]   M. G. Kendall and A. Stuart, *The Advanced Theory of Statistics*. vol. 1,
        New York: Hafner Publishing Co., 1963.

[5]   A. N. Kolmogorov and S. V. Fomin, *Elements of the Theory of Functions
        and the Functional Analysis*. vol. 2. Baltimore: Graylock Press,
        1961.

[6]   S. R. Liberty, "Characteristic Functions of LQG Control," Ph.D.
        dissertation, Dept. of Elec. Engrg., Univ. of Notre Dame, Notre
        Dame, Ind., August 1971.

[7]   S. R. Liberty and R. C. Hartwig, "On the Essential Quadratic
        Nature of LQG Control-Performance Measure Cumulants," Information
        and Control, vol. 32, pp. 276-305, November 1976.

[8]   I. B. Rhodes, "A Tutorial Introduction to Estimation and Filtering,"
        IEEE Trans. on Automat. Contr., vol. AC-16, no.6 pp.688-706,
        December, 1971.

[9]   M. K. Sain, "Relative Costs of Mean and Variance Control for a
        Class of Linear, Noisy Systems," in *Proc. 3rd Annual Allerton
        Conf. on Circuit and System Theory*, pp. 121-129, October 1965.

[10]   M. K. Sain, "Control of Linear Systems According to the Minimal
        Variance Criterion--A New Approach to the Disturbance Problem,"
        IEEE Trans. Automat. Contr., vol. AC-11, pp. 118-121, January 1966.

[11]   M. K. Sain, "Performance Moment Recursions with Application to
        Equalizer Control Laws," in *Proc. 5th Annual Allerton Conf. on
        Circuit and System Theory*, pp. 327-336, October 1967.

[12]   M. K. Sain and S. R. Liberty, "Some New Computational Results
        for Minimum Variance Controllers," in *Proc. 7th Annual Allerton
        Conf. on Circuit and System Theory*, pp. 408-417, October 1969.

[13]   M. K. Sain and S. R. Liberty, "Performance Measure Densities for
        a Class of LQG Control Systems," IEEE Trans. Automat. Contr.,
        vol. AC-16, pp. 431-439, October 1971.

[14]   D. D. Sworder and L. L. Choi, "Stationary Cost Densities for
        Optimally Controlled Stochastic Systems," IEEE Trans. on
        Automat. Contr., vol. AC-21, no. 4, pp. 492-499, August 1976.

[15]   E. Tse, "On the Optimal Control of Stochastic Linear Systems,"
        IEEE Trans. Automat. Contr., vol. AC-16, pp. 776-785, December 1971.

[16]   W. M Wonham, "On the Separation Theorem of Stochastic Control,"
        SIAM J. Control, vol. 6 no. 2, pp. 312-325, 1968.

III. APPENDIX B

INITIAL STUDIES OF RICCATI EQUATIONS ARISING IN STOCHASTIC

LINEAR SYSTEM THEORY

by

MONG LING YAO, B.S. in E.E.

A THESIS

IN

ELECTRICAL ENGINEERING

Submitted to the Graduate Faculty
of Texas Tech University in
Partial Fulfillment of
the Requirements for
the Degree of

MASTER OF SCIENCE

IN

ELECTRICAL ENGINEERING

Approved

Chairman of the Committee

Accepted

Dean of the Graduate School

December 1977

ACKNOWLEDGEMENTS

I am deeply indebted to Dr. Stanley R. Liberty for his suggestion of this research topic and for his patience and guidance throughout; to Dr. Wayne T. Ford for his helpful assistance and suggestions which aided in the analytic portion of this work; to Dr. Kwong S. Chao and Dr. Benjamin S. Duran for their consideration and criticisms in the preparation of this thesis.

TABLE OF CONTENTS

LIST OF TABLES

Table

LIST OF FIGURES

CHAPTER I

INTRODUCTION

A new class of matrix differential equations has arisen in the context of Stochastic linear system design. These equations are nonlinear and are parameterized such that the classical Riccati equation is imbedded in the class. Because of this, the new class of equations is referred to as "Riccati type". The work presented herein is the first investigation of some of the properties of these equations and the systems in which they are utilized.

Chapter II contains a summary of the derivation of these new equations. In Chapter III, the positive definite and symmetric properties associated with this new class of matrix differential equations are shown. Chapter IV contains steady state results for the scalar case. In Chapter V, three numerical examples of linear estimation and control are presented in which the Riccati equations are utilized as design tools. Conclusions and suggestions for further research are contained in Chapter VI.

CHAPTER II

LINEAR STOCHASTIC SYSTEMS AND RICCATI EQUATIONS

An interesting class of matrix differential equations arises in the context of determining complete statistical descriptions of integral quadratic forms in random processes generated by linear dynamical systems. These systems operate on Gaussian-white noise to produce vector valued Gauss-Markov processes. The derivation of these equations is contained in [1] and the results are summarized here.

Consider the stochastic linear dynamical system

$$\dot{x}(t) = F(t)x(t) + G(t)\xi(t), \qquad (2-1)$$

where the state $x(t) \in R^n$ (the n-fold cartesian produces of the real line) and the noise $\xi(t) \in R^m$. The process $\xi(t)$ is assumed to be Gaussian white with zero mean and covariance kernel.

$$E\{\xi(t)\xi^T(t)\} = \Xi(t)\delta(t-\tau). \qquad (2-2)$$

The initial state for (2-1), $x(t_o)$, is assumed to be Gaussian with

$$E\{x(t_o)\} = x_o, \qquad (2-3)$$

and

$$E\{[x(t_o)-x_o][x^T(t_o)-x_o^T]\} = P_o ; \qquad (2-4)$$

and (2-1) is defined on a finite interval $[t_o, t_f]$.

2

Attached to (2-1) is a performance measure, J, defined by

$$J = x^T(t_f)Sx(t_f) + \int_{t_o}^{t_f} x^T(t)N(t)x(t)dt , \qquad (2-5)$$

where S and N(t) are positive semi-definite. In [1] Liberty and Hartwig show that the statistics of J (in particular the cumulants of J) can be expressed explicitly in terms of a countable set of matrix variables satisfying simultaneous differential equations. The particular matrix variables of interest in this work are the H variables of [1] which evolve according to

$$\dot{H}(\beta,1) = -F^T(\beta)H(\beta,1)-H(\beta,1)F(\beta)-N(\beta), \quad \beta\epsilon[t_o,t_f],$$
$$(2-6)$$

and

$$\dot{H}(\beta,k) = -F^T(\beta)H(\beta,k)-H(\beta,k)F(\beta)$$
$$(2-7)$$
$$- \sum_{j=1}^{k-1} H(\beta,j)G(\beta)Q(\beta)G(\beta)H(\beta,k-j) \quad \beta\epsilon[t_o,t_f]$$
$$k = 2,3,4,\dots.$$

with boundary conditions

$$H(t_f,k) = S, \qquad (2-8)$$

and

$$H(t_f,k) = 0, \qquad k= 2,3\dots, \qquad (2-9)$$

It is shown in [1] that the kth cumulant of J contains a term of the form $x_o^T H(t_o,k)x_o$.

The class of Riccati equations of interest in this work arose in the work of Hartwig [2] in the following way. Consider a stochastic linear control system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + G(t)\xi(t) \qquad (2-10)$$

where the control action, $u(t)\epsilon R^p$, is to be selected such that system performance

$$J = x^T(t_f)Sx(t_f) + \int_{t_o}^{t_f} [x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)]\,dt \qquad (2-11)$$

is good in some statistical sense. In (2-11), $Q(t)$ is positive semi-definite and $R(t)$ is positive definite. It is well-known that if $u(t)$ is chosen as

$$u(t) = -R^{-1}(t)B(t)K(t)x(t) , \qquad (2-12)$$

where $K(t)$ satisfies the Riccati equation

$$\dot{K}(t) = -K(t)A(t) - A^T(t)K(t) + K(t)B(t)R^{-1}(t)B^T(t)K(t) - Q(t), \qquad (2-13)$$

then the expected value of J is minimized and contains a term that is quadratic in the initial state mean given by $x_o^T K(t_o)x_o$.

It should also be noted that substitution of (2-12) into (2-11) and (2-9) identifies

$$F(t) = A(t) - B(t)R^{-1}(t)B^T(t)K(t) \qquad (2-14)$$

and

$$N(t) = Q(t) + K(t)B(t)R^{-1}(t)B^T(t)K(t). \qquad (2\text{-}15)$$

It was also observed in [2] that under the conditions of (2-12) and (2-13)

$$K(t_o) = H(t_o,1) \qquad (2\text{-}16)$$

Indeed if $U(t)$ had been selected as

$$u(t) = -R^{-1}(t)B^T(t)H(t,1)x(t) \qquad (2\text{-}17)$$

then substitution of (2-17) into (2-10) and (2-12) would lead to

$$F(t) = A(t) - B(t)R^{-1}(t)B^T(t)H(t,1) \qquad (2\text{-}18)$$

and

$$N(t) = Q(t) + H(t,1)B(t)R^{-1}(t)B^T(t)H(t,1). \qquad (2\text{-}19)$$

Substitution of (2-18) and (2-19) into (2-6) yields

$$\begin{aligned}\dot{H}(t,1) = -\,H(t,1)A(t) - A^T(t)H(t,1) \\ +\, H(t,1)B(t)R^{-1}(t)B^T(t,1) - Q(t)\end{aligned} \qquad (2\text{-}20)$$

If one were to form a linear combination of cumulants of J then such a combination would contain a term of the form

$$x_o^T[\alpha_1 H(t_o,1) + \alpha_2 H(t_o,2) + \ldots\ldots\ldots\alpha_\ell H(t_o,\ell)] \qquad (2\text{-}21)$$

In [1] Hartwig argues that if one were interested in
selecting $u(t)$ such that several statistics of J would be
affected in a prescribed way then it might be reasonable to
select $u(t)$ as

$$u(t) = - R^{-1}(t)B^T(t)[\alpha_1 H(t,1)$$
$$\quad + \alpha_2 H(t,2) + \ldots \ldots \alpha_\ell H(t,\ell)]x(t). \tag{2-22}$$

He subsequently demonstrated by several examples that
this conjecture was reasonable and good performance could be
achieved in this sense. Substitution of (2-22) into (2-6)
and (2-7) yields the class of Riccati equations of interest
in this work.

For this initial study of the properties and character-
istics of these equations only the first two equations are
studied. Without loss of generality α_1 is selected to be
1 and α_2 is replaced by α.

$$\dot{H}(t,1) = - A^T(t)H(t,1) - H(t,1)A(t) - Q(t)$$
$$\quad - \alpha^2 H(t,2)B(t)R^{-1}(t)B^T(t)H(t,2) \tag{2-23}$$
$$\quad + H(t,1)B(t)R^{-1}(t)B^T(t)H(t,1) \ ,$$

and

$$\dot{H}(t,2) = -A^T(t)H(t,2) - H(t,2)A(t) + H(t,1)B(t)R^{-1}(t)B^T(t)H(t,2)$$
$$\quad + H(t,2)B(t)R^{-1}(t)B^T(t)H(t,1)$$
$$\quad + 2\alpha H(t,2)B(t)R^{-1}(t)B^T(t)H(t,2) - H(t,1)\Xi(t)H(t,1),$$

with the boundary conditions (2-8) and (2-9). $\qquad$ (2-24)

CHAPTER III

MATRIX RESULTS

III-A Eigenvalue Trajectories

For a first look at the properties of (2-23) and (2-24) let

$$[t_o, t_f] = [0, 10] \qquad (3-1)$$

and select the coefficients

$$A(t) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}, \qquad (3-2)$$

$$Q(t) = 0 \text{ (the zero matrix)}, \qquad (3-3)$$

$$B(t) = R(t) = \Xi(t) = I \text{ (the identity matrix)}, \qquad (3-4)$$

and final values

$$H(10,1) = 1, \qquad (3-5)$$

$$H(10,2) = 0. \qquad (3-6)$$

Note that the Pair $[A,B]$ is controllable. Figures (3.1) to (3.4) contain the trajectories of the eigenvalues of $H(t,1)$ and $H(t,2)$ for $\alpha = 0, 1, 2$ and 4 respectively. These eigenvalues of $H(t,1)$ are designated by $\lambda_{i1}(t)$, $i = 1, 2, 3$.

7

Those of $H(t,2)$ are correspondingly designated by $\lambda_{i2}(t)$, $i = 1, 2, 3$. Note that for small values of α, the trajectories are well-behaved. Indeed, selection of an arbitrarily large time interval will result in steady state solutions in negative time. However, for larger values of α, it appears that steady state solution may not exist and the trajectories may be unbounded on the half line.

Another interesting property is that the trajectories are non-negative for all t. It is interesting to note that this is a general property of (2-23) and (2-24) which can easily be shown.

III-B Non-negative Definiteness

It is claimed that the matrix solutions to (2-23) and (2-24) with the final values (2-8) and (2-9) are positive semi-definite (non-negative definite) for all t. To see this select any $e \epsilon R^n$, $e \neq \theta$ (the null vector). Place $H(t,2)$ in a quadratic form in e as

$$
\begin{aligned}
e^T \dot{H}_2 e = &- e^T A(t) H_2 e - e^T H_2 A^T(t) e \\
&+ e^T H_1 B(t) R^{-1}(t) B^T(t) H_2 e \\
&+ e^T H_2 B(t) R^{-1}(t) B^T(t) H_1 e \\
&+ 2\alpha e^T H_2 B(t) R^{-1}(t) B^T(t) H_2 e - e^T H_1 \Xi H_1 e
\end{aligned}
\tag{3-7}
$$

where H_2 means $H(t,2)$ and H_1 means $H(t,1)$.

It is easy to see that if $R(t)$, $Q(t)$, $\Xi(t)$ and the final value are symmetric then the matrices $H(t,1)$ and $H(t,2)$ are symmetric for all t. If the symmetric H is ever non-negative definite then a non matrix H_2 can always be found such that

$$H_2 = \tilde{H}_2^T \tilde{H}_2 \tag{3-8}$$

Then

$$e^T H_2 e = e^T \tilde{H}_2^T \tilde{H}_2 e = ||\tilde{H}_2 e||^2 \tag{3-9}$$

which is equal to zero if and only if $\tilde{H}_2 e$ is the zero vector. Now consider the H_2 trajectory backward in time from $t = t_f$. Initially (finally) $H(t_f,2)$ is the zero matrix so from (3-7)

$$e^T H(t_f,2) e = -e^T S \Xi(t_f) S e \leq 0 \tag{3-10}$$

since $\Xi(t_f)$ is non-negative definite. This means that as the trajectory of H_2 is followed backward in time from $t = t_f$, H_2 initially becomes no less negative. Under reasonable smoothness assumptions on the coefficients of (2-24), the solution, H_2, is continuous implying that the scalar quantity $e^T H(t,2) e$ is also continuous. Now consider two points in time $t_1 < t_2$. It follows that in order for $e^T H(t,2) e$ to be negative, given that $e^T H(t_2,2) e$ is positive, there must exist a t_3 with $t_1 < t_3 < t_2$ such that

$$e^T H(t_3,2) e = 0, \tag{3-11}$$

But from previous arguments applied at $t = t_f$, whenever (3-11) is satisfied

$$e^T H(t_3, 2) e \leq 0. \qquad (3-12)$$

Thus $e^T H_2 e$ equal to zero is a reflective barrier which cannot be penetrated. Thus H_2 is always non-negative definite. Similar arguments will also show that H_1 is non-negative definite.

CHAPTER IV

SCALAR RESULTS

From this point on the sense of time in equations (2-23) and (2-24) will be reversed. This is equivalent to negating the right sides of (2-23) and (2-24). With this change, the equations become forward time equations and the content of this chapter is devoted to examination of steady state solution questions for scalar H_1 and H_2 with constant coefficients. Rewriting (2-23) and (2-24) as scalar forward time equations with constant coefficients, one obtains

$$\dot{H}_1 = 2aH_1 + \alpha^2 dH_2^2 - dH_1^2 + q , \qquad (4-1)$$

and

$$\dot{H}_2 = 2aH_2 - 2dH_1H_2 - 2\alpha dH_2^2 + \Xi H_1^2 , \qquad (4-2)$$

where for simplicity

$$d = b^2 r^{-1} . \qquad (4-3)$$

Define new dependent variables, V and W by

$$W \triangleq H_1 e^{-2at} , \qquad (4-4)$$

and

$$V \triangleq H_2 e^{-2at}. \qquad (4-5)$$

Substituting for H_1 and H_2 in (4-1) and (4-2) yields

$$\dot{W} = (-dW^2 + \alpha^2 dV^2 + qe^{-4at})e^{2at} , \qquad (4-6)$$

11

and

$$\dot{V} = (-2dVW - 2d\alpha V^2 + \Xi W^2)e^{2at}. \tag{4-7}$$

Define the new independent variable x by

$$x(t) \triangleq \int_0^t e^{2a\tau}d\tau = \frac{1}{2a}[e^{2at} - 1]. \tag{4-8}$$

Then

$$e^{2at} = 2ax(t) + 1, \tag{4-9}$$

next let

$$W' = dW/dx = \frac{\partial W}{\partial t}(dt/dx) = \dot{W}e^{-2at}, \tag{4-10}$$

and,

$$V' = dV/dx = \frac{\partial V}{\partial t}(dt/dx) = \dot{V}e^{-2at}. \tag{4-11}$$

Now, rewrite (4-10) and (4-11) in the form

$$W' = -dW^2 + \alpha^2 dV^2 + q(2ax + 1)^{-2} \tag{4-12}$$

$$V' = -2dVW - 2\alpha dV^2 + \Xi W^2, \tag{4-13}$$

subject to the initial conditions

$$W(0) = W(x)\Big|_{x=0} = H_1(t)e^{-2at}\Big|_{t=0} = H_1(0) = S, \tag{4-14}$$

and

$$V(0) = V(x)\Big|_{x=0} = H_2(t)e^{-2at}\Big|_{t=0} = H_2(0) = 0. \tag{4-15}$$

Define a new dependent variable U by

$$U \triangleq V/W. \tag{4-16}$$

Then

$$U(0) = V(0)/W(0) = 0, \tag{4-17}$$

and

$$U = UW. \tag{4-18}$$

It follows that

$$U' = (-1/W)[2\alpha dU^2 W - \Xi W_-^2 + dW^2 U$$

$$+ \alpha^2 dU^3 W^2 + qU(2ax + 1)^{-2}]. \tag{4-19}$$

Equations (4-12) and (4-19) with boundary conditions (4-14) and (4-17) provide a mechanism for examining steady state properties of (4-1) and (4-2). Letting $S = 1$, Figures (4.1) through (4.5) contain U-W phase plane plots for several different values of a, d, q and Ξ. Interpretation of these plots is enhanced by observing that $(\Xi/4d, W)$ with W arbitrary is an equilibrium point of (4-12) and (4-19) in the V-W phase plane if $\alpha = 4d/\Xi$.

On all of these plots for $0 \leq \alpha < 4d/\Xi$, the phase plane trajectories approach $W = 0$ implying the existence of a steady state solution for H_1. The value of U approached yields the corresponding steady state solution for H_2 since

$$U = H_2/H_1. \tag{4-20}$$

For $\alpha = 4d/\Xi$ no steady state is reached for H_1 but the time trajectories of H_1 and H_2 are in proportion as

$t \to \infty$ by relation $H(t,2) \to (\Xi/4d)H(t,1)$. That is, U approaches $\Xi/4d$. For $\alpha > 4d/\Xi$, numerical instability occurs.

Thus it may be concluded that steady state solutions to the Riccati equations studied herein may exist but only for limited ranges of the parameter α.

CHAPTER V

SYSTEM EXAMPLES

Now that some of the properties of the Riccati equations have been observed, it would be interesting to see the properties of systems that contain these Riccati equation solutions as parameters. In this chapter one example of linear estimators and two examples of linear feedback controllers that contain Riccati equation parameters are presented.

In both example classes, the gains that are normally a function of the classical Riccati equation are replaced by gains with the term replaced by the weighted sum of the two Riccati solutions studied herein. The weighting parameter is, of course, the parameter α, that appears in the actual equations. Note that when $\alpha = 0$ these new classes of controllers and estimators reduce to the classical cases. The example studies here are carried out primarily as a function of the parameter α.

<u>Example 1</u>:   Filtered estimate of nonlinear process

Consider the nonlinear system

$$\dot{x}(t) = \sin(x(t)) , \qquad\qquad (5-1)$$

with measurement

$$y(t) = x(t) + \theta(t) , \qquad\qquad (5-2)$$

15

where $\theta(t)$ is zero-mean and white with unity covariance. The initial condition for (5-1) is random with assumed mean of zero and unity covariance.

In this example it is desired to estimate $x(t)$ based upon observations $y(\tau)$, $0 \leq \tau \leq t$. To accomplish this the system (5-1) is first linearized to give a linear model

$$\dot{\hat{x}}(t) = \hat{x}(t) , \tag{5-3}$$

$$\hat{y}(t) = \hat{x}(t) + \theta(t). \tag{5-4}$$

Based upon this model, which is a good approximation only for small $x(t)$, a linear filter is designed utilizing the classical structure shown in Figure (5.1) with the filter gain, $P(t)$, selected as

$$P(t) = H_1(t) + \alpha H_2(t) \tag{5-5}$$

where $H_1(t)$ and $H_2(t)$ are solutions to

$$\dot{H}_1 = 2H_1 + \alpha^2 H_2^2 - H_1^2 , \tag{5-6}$$

and

$$\dot{H}_2 = 2H_2 - 2H_1 H_2 - 2\alpha H_2^2 + H_1^2 , \tag{5-7}$$

with

$$H_1(0) = 1 , \tag{5-8}$$

and

$$H_2(0) = 0 . \tag{5-9}$$

Note that (5-6) and (5-7) are the duals in the traditional control/filtering sense of (2-23) and (2-24). There is one subtlety, however, in that there is no dual for the $\Xi(t)$ coefficient in (2-24). Consequently it becomes a design parameter and has arbitrarily been set to unity in this example. It should be noted that if this coefficient is set to zero, then H_2 will be identically zero and solution of (2-23) and (2-24) for any value of α is equivalent to solution with $\alpha = 0$. It is interesting to note that the steady state value for $x(t)$ in (5-1) is π and so (5-3) is not a good approximation. Despite this the linear filter performs quite well. In Figure (5.2) the actual state trajectory evolving from a randomly selected initial condition of $x_o = 0.8$ and filtered estimate trajectories for $\alpha = 0$, 1, and 2 are shown. Note that for $\alpha = 0$ which is the classical Kalman filter, the estimate is not as good as for the cases $\alpha = 1$ and 2. In these cases, however, there tends to be more oscillatory behavior.

Example 2:   A Second Order Linear Regulator
                Control System

    In this example consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad\qquad (5\text{-}10)$$

with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{5-11}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5-12}$$

and

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{5-13}$$

The control action u is selected to be

$$u(t) = -R^{-1}B^T[H_1 + \alpha H_2]x(t) \tag{5-14}$$

where H_1 and H_2 are the steady state solutions to equations (2-23) and (2-24) with

$$\Xi = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5-16}$$

and

$$R = 1. \tag{5-17}$$

In Figures (5.3) and (5.4) the state trajectories of the closed loop system are shown for $\alpha = 0, 5$ and 9. This example first appeared in [2] where the system was noisy and Ξ was the noise covariance. In [2] control laws similar to those here were selected, but only over a finite time

interval. It was demonstrated in [2] that these controllers were good in a second order statistical sense. The question addressed here is that of the asymptotic stability of (5-10) under the feedback (5-14). After all, in order for the control laws suggested by Hartwig in [2] to be acceptable, they must not only have good statistical properties, but should also stabilize the system.

Note again as in Example 1 that when $\alpha = 0$ the feedback gain collapses on a classical structure. In this case it is the classical linear regulator which is known to be asymptotically stable. As can be seen from the state trajectories in Figures (5.3) and (5.4). The closed loop system is also stable for the non zero values of α. Indeed, it appears from these time domain pictures that for larger the system is more stable than for the case $\alpha = 0$.

To obtain a complete picture of the stability of the feedback system look at Figure (5.5) where the loci of the closed loop poles of the system are plotted in the complex plane as a function of α.

Table I contains several classical second order system parameters ξ, w_n, T_d and % overshoot as a function of α. It is most interesting to note that for $\alpha > 9$ numerical instability arose in attempting to find steady state solutions to the Riccati equations so instability of the closed loop control system was never attained.

<u>Example 3</u>:   A Third Order Linear Regulator
                      Control System

A last example of third order will again demonstrate
similar stability properties as a function of α. The system
is the two-vehicle problem described in [3]. That is a
linear system with

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \qquad (5\text{-}18)$$

and

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad (5\text{-}19)$$

The feedback control law is again chosen as in (5-4) with

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (5\text{-}20)$$

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (5\text{-}21)$$

and Ξ chosen first as

$$\Xi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \qquad (5\text{-}22)$$

State trajectories for this case with

$$x(0) = \begin{bmatrix} -2.5 \\ -4.3 \\ 0 \end{bmatrix} \tag{5-23}$$

appear in Figures (5.6), (5.7) and (5.8) for $\alpha =$, 0.7 and 1.2.   Changing Ξ to

$$\Xi = \begin{bmatrix} 5.0 & 0 & 0 \\ 0 & 5.0 & 0 \\ 0 & 0 & 5.0 \end{bmatrix} \tag{5-24}$$

yields the trajectories in Figures (5.9) , (5.10) and (5.11). Corresponding pole loci for the closed loop systems are shown in Figures (5.12) and (5.13) with accompanying parameters in Tables II and III.

CONCLUSIONS

Future research on the new class of Riccati equations should address the question of global existence of solutions. Numerical behavior of the equations indicates that global existence may be subject to the value of the α parameter. The specific relationship between open-loop system properties, the Riccati equations and closed loop system performance needs to be analytically explored. The relationship between the value of α and the relative stability of a closed loop control system designed via these equations is only one example of this.

A detailed study of estimators (filters in particular) designed via the new equations needs to be carried out. Whether or not the properties observed here, in nonlinear application are general, is a completely open question.

A full understanding of the duality between the filtering and control contexts needs to be developed.

LIST OF REFERENCES

[1]   Liberty, S. R. and Hartwig, R. C., "On the Essential
       Quadratic Nature of LQG Control-Performance Measure
       Cumulants," Information Control, vol. 32, no. 3,
       pp. 276-305, November 1976.

[2]   Hartwig, R. C., "Design of Stochastic Linear Control
       Systems According to Cumulant Based Performance
       Criteria," Ph.D. Dissertation, Texas Tech University,
       Lubbock, Texas, May 1976.

[3]   Levine, W. S., and Athans, M., "On the Optimal Error
       Regulation of a String of Moving Vehicles," IEEE
       Trans. on Automatic Control, vol. AS-11, pp. 355-361,
       July 1966.

APPENDIX A

ξ, Wn, Td, and % Overshoot

24

TABLE I

ξ, Wn, Td (The Period of Oscillation) and % Overshoot as the Function of α for Example 3

α	λ_1 , λ_2		ξ	ωn	Td	% overshoot
0.0	-0.8660	±0.5000j	0.866025	1.0	12.57	0.43
0.1	-0.8941	±0.4477j	0.894100	1.0	14.03	0.19
0.2	-0.8210	±0.3894j	0.903400	0.9087	16.13	0.13
0.3	-0.9470	±0.3211j	0.947000	0.9999	19.56	0.095
0.4	-0.9723	±0.2335j	0.972350	0.9999	26.90	0.0002
0.5	-0.9971	±0.0747j	0.997198	0.9999	83.99	6.49×10^{-17}
0.6	-0.8122	,-1.0216	1.006580			
0.7	-0.7393	,-1.0459	1.015000			
1.0	-0.6180	,-1.1179	1.044220			
2.0	-0.4353	,-1.3662	1.168000			
3.0	-0.3364	,-1.6540	1.334000			
4.0	-0.2652	,-2.0175	1.560000			
5.0	-0.2077	,-2.5110	1.880000			
6.0	-0.1589	,-3.2418	2.368500			
7.0	-0.1134	,-4.4629	3.215400			
8.0	-0.0721	,-6.9631	4.961820			
9.0	-0.0318	,-15.085	10.91312			

TABLE II

ξ, Wn, Td and % Overshoot as the Function of α for

Example 3 with the State Noise Covariance

Equal to Unity

α	λ_1 , λ_2	ξ	ωn	Td	% overshoot
0.0	-2.19800 ±1.40900j	0.8418	2.6108	4.460	0.7400
0.1	-2.43550 ±1.48540j	0.8537	2.8520	4.230	0.3800
0.2	-2.63018 ±1.51330j	0.8667	3.0344	4.150	0.4300
0.3	-2.80475 ±1.49225j	0.8828	3.1770	4.210	0.2700
0.4	-2.96051 ±1.40080j	0.9039	3.2752	4.490	0.1300
0.5	-3.07645 ±1.17901j	0.9337	3.2946	5.330	0.0275
0.6	-3.01460 ±0.55609j	0.9834	3.0654	62.255	1.77×10^{-39}

TABLE III

ξ, Wn, Td and % Overshoot as the Function of α for Example 3 with the State Noise Covarinace Equal to 5I.

α	λ_1 , λ_2	ξ	ωn	Td	% overshoot
0.0	-2.1980 $\pm$1.4090j	0.841870	2.610800	4.460	0.74000
0.1	-3.0760 $\pm$1.1790j	0.833759	3.294200	5.330	0.02750
0.11	-3.0930 $\pm$0.9651j	0.954600	3.240000	6.509	0.00424
0.12	-3.0145 $\pm$0.5560j	0.983410	3.065300	11.300	4.01×10^{-6}
0.122	-2.9586 $\pm$0.4000j	0.990784	2.985517	15.700	8.098×10^{-9}
0.123	-2.9106 $\pm$0.2878j	0.895146	2.924790	21.830	1.59×10^{-12}
0.124	-2.7920 $\pm$0.05233	0.999820	2.782490	1.2×10^2	1.94×10^{-71}

APPENDIX B

Time Domain Trajectories, Phase Plane and

Closed Loop Pole Locations

28
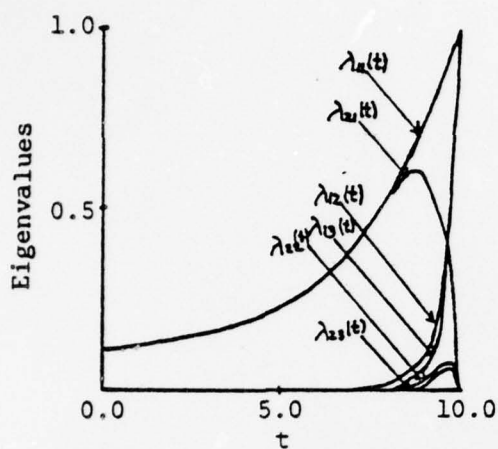
Fig. (1.1) The eigenvalue trajectories for equations (2-23) and (2-24) for the conditions (3-1) to (3-5) $\alpha = 0$.
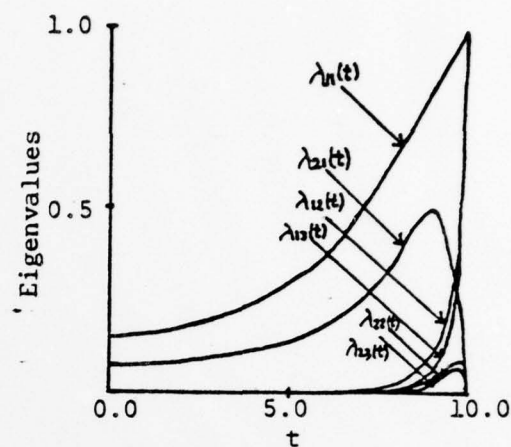
Fig. (1.2) The eigenvalue trajectories for equations (2-23) and (2-24) for the conditions (3-1) to (3-5) $\alpha = 1$.
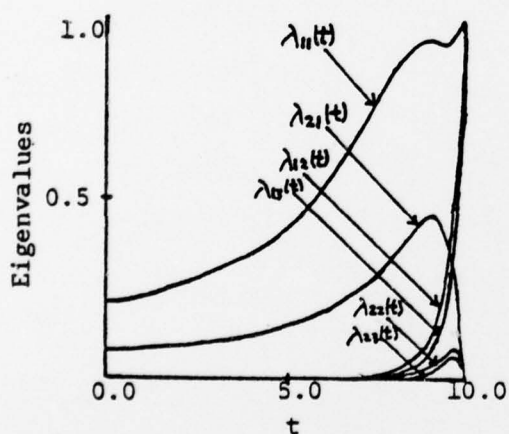
Fig. (3.3) The eigenvalue trajectories for equations (2-23) and (2-24) for the conditions (3-1) to (3-5) $\alpha = 2$.
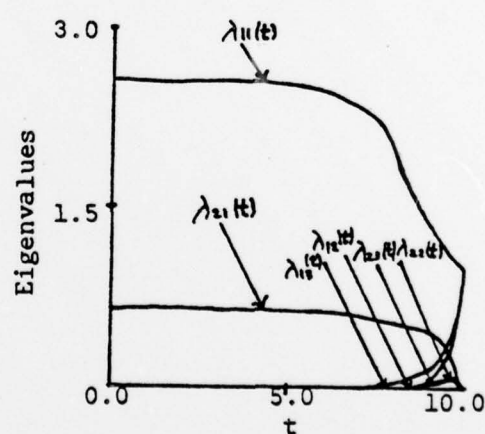
Fig. (3.4) The eigenvalue trajectories for equations (2-23) and (2-24) for the conditions (3-1) to (3-5) $\alpha = 4$.
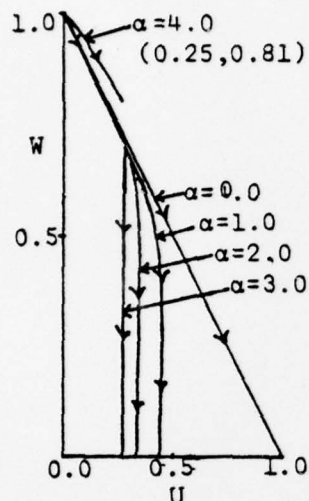
Fig. (4.1) The U-W phase plane plots for equations (4-12) and (4-19) with a=1, d=1, q=0, Ξ=1.
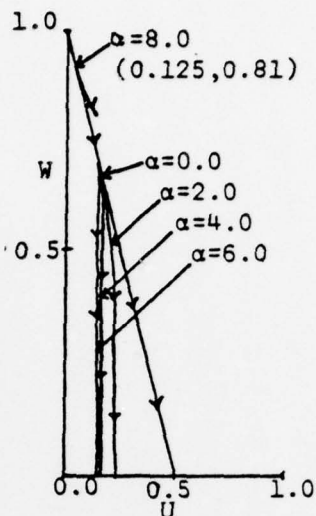
Fig. (4.2) The U-W phase plane plots for equations (4-12) and (4-19) with a=1, d=1, q=0, Ξ=0.5.
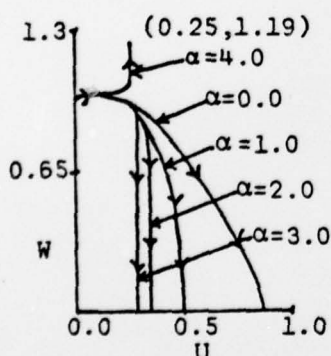
Fig. (4.3) The U-W phase plane plots for equations (4-12) and (4-19) with a=1, d=1, q=1, Ξ=1.
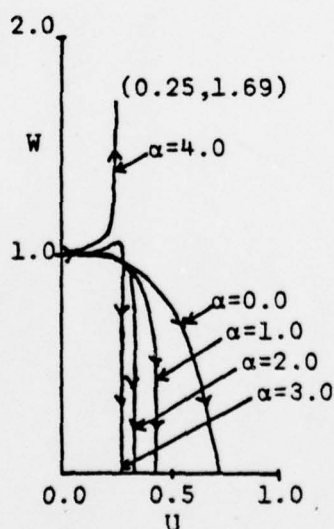
Fig. (4.4) The U-W phase plane plots for equations (4-12) and (4-19) with a=1, d=0.5, q=0.6, Ξ=0.5.
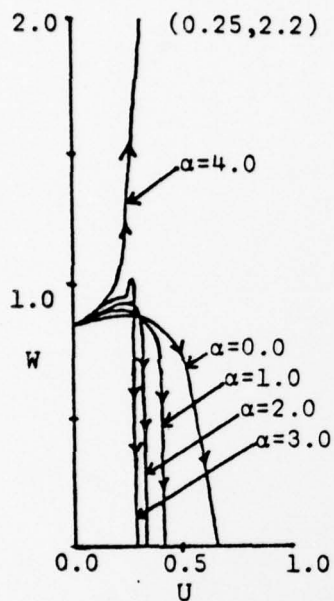
Fig. (4.5) The U-W phase plane plots for equations (4-12) and (4-19) with a=0.2, d=0.5, q=0.8, Ξ =0.5.
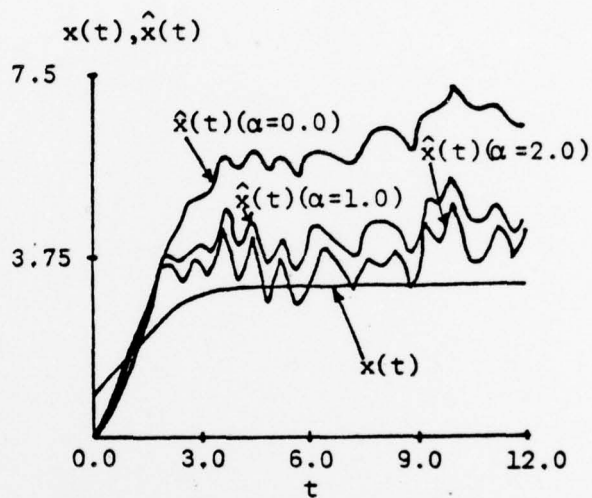
Fig. (4.6) The actual state and the measurement states with different α for Example 1.
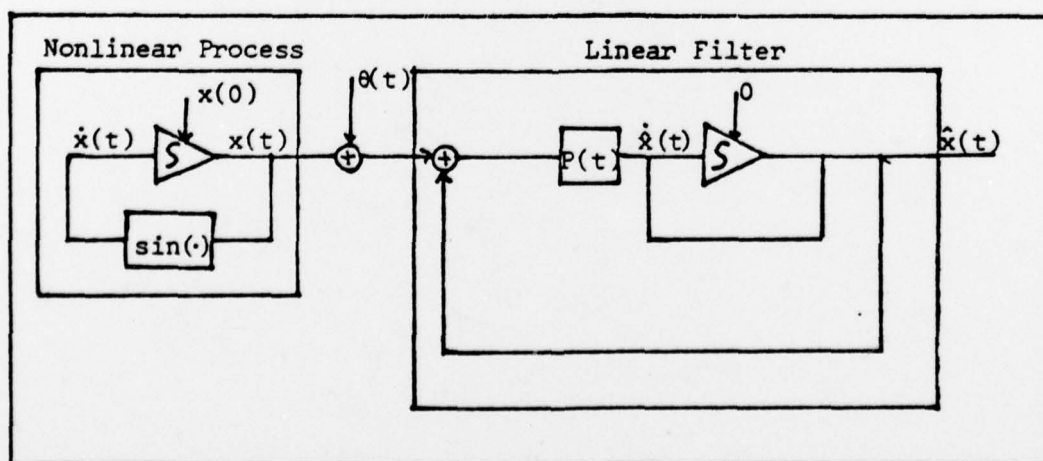
Fig. (5.1) Block Diagram of Example 1
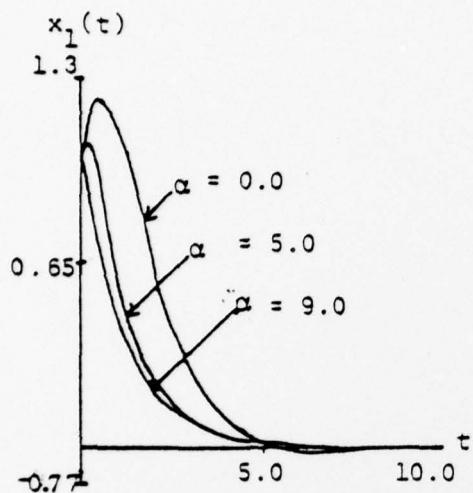
Fig. (5.3) Time domain
trajectories of x_1 in
Example 2.
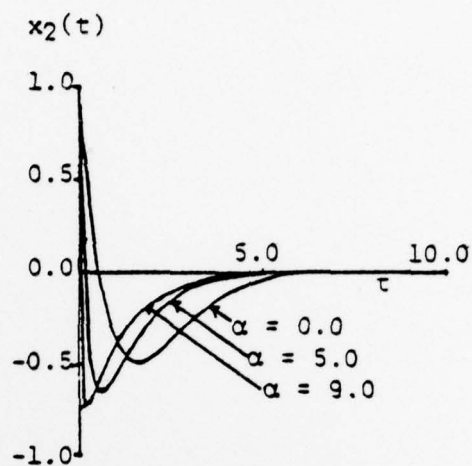
Fig. (5.4) Time domain
trajectories of x_2 in
Example 3.
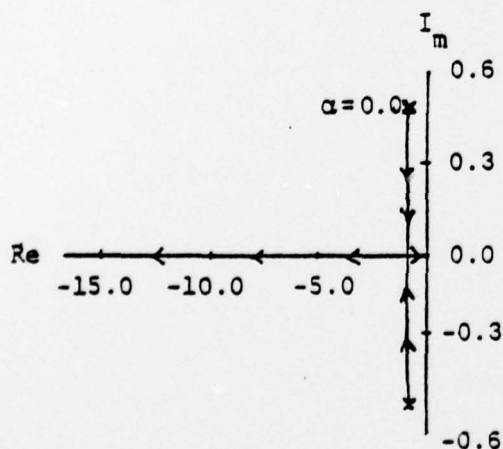
Fig. (5.5) Closed loop
pole locations as a
function of α for
Example 2.
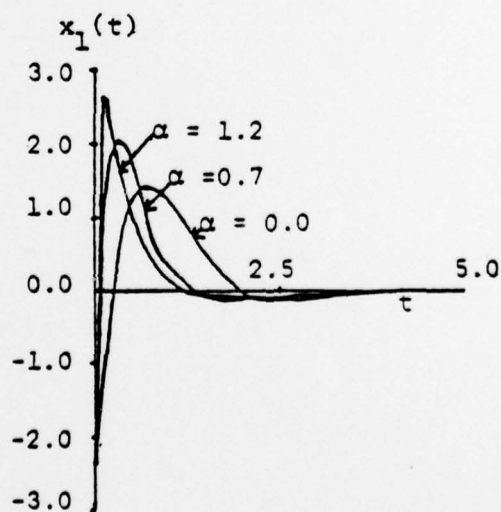
Fig. (5.6) Time domain
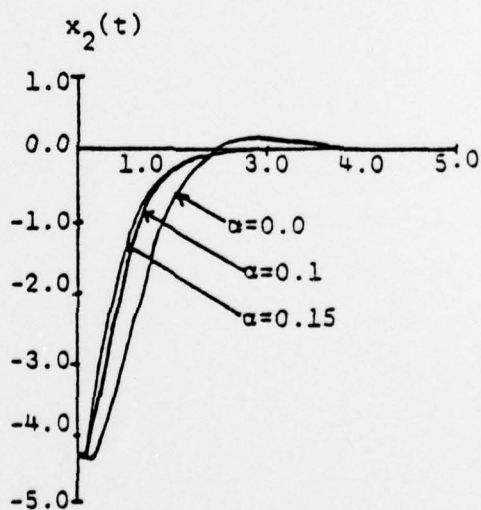trajectories of x_1 in
Example 3 with $\Xi = I$.
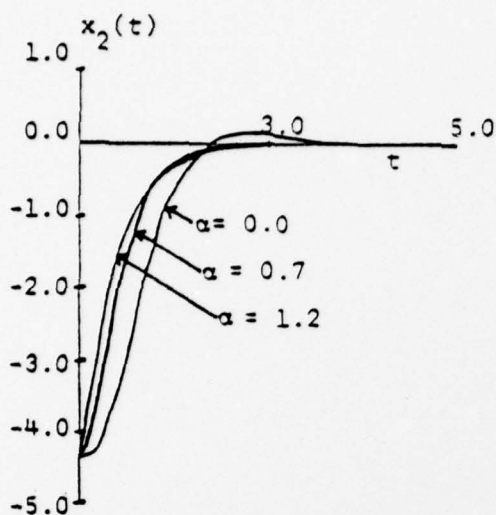
Fig. (5.7) Time domain
Trajectories of x_2 in
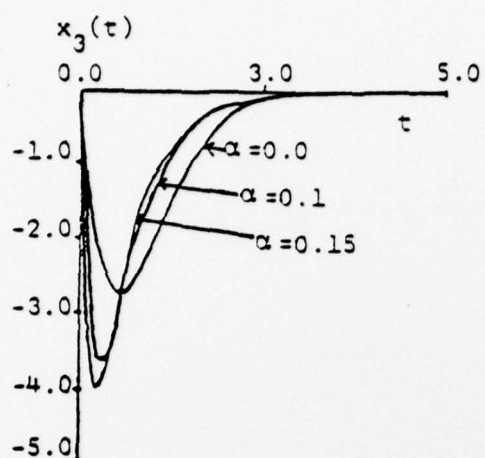Example 3 with $\Xi = 2I$.
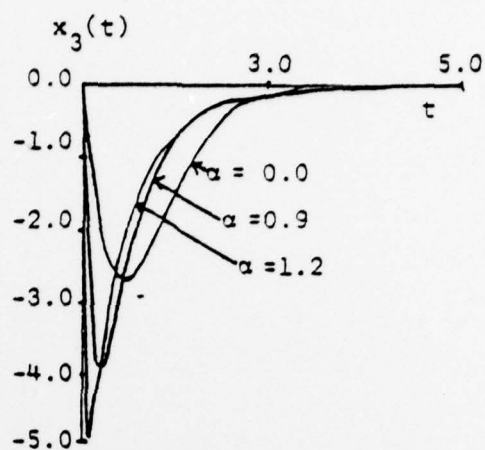
Fig. (5.8) Time domain
trajectories of x_3 in
Example 3 with $\Xi = 3I$.
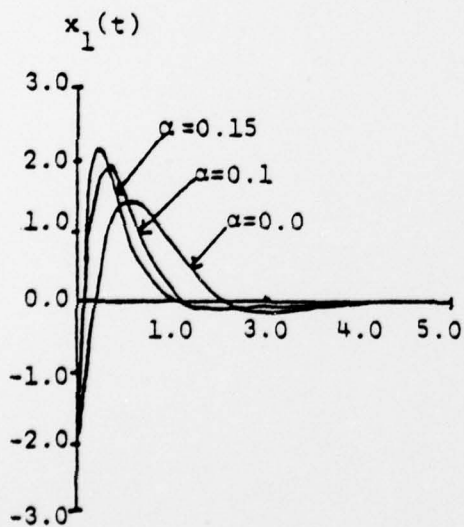
Fig. (5.9) Time domain
trajectories of x_1 in
Example 3 with $\Xi = 5I$.

Fig. (5.10) Time domain
trajectories of x_2 in
Example 3 with $\Xi = 5I$.

Fig. (5.11) Time domain trajectories of x_3 in Example 3 with $\Xi = 5I$.
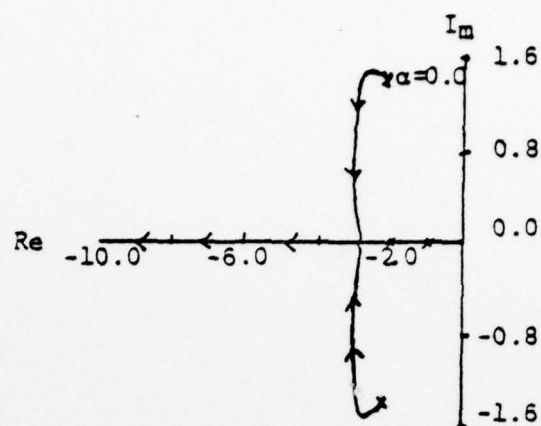

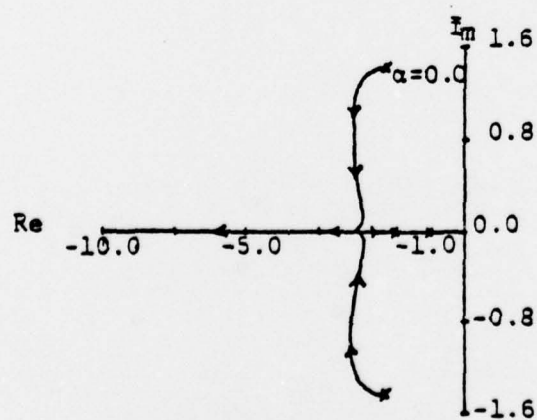Fig. (5.12) Closed loop pole locations as a function of α for Example 3 with $\Xi = I$.

Fig. (5.13) Closed loop pole locations as a function of α for Example 3 with $\Xi = 5I$.